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The 2-Transitive Ovoids

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1. INTRODUCTION

An *ovoid* $\mathcal{O}$ in a classical polar space is a set of singular points such that every maximal totally singular subspace contains just one point in $\mathcal{O}$. Ovoids are intimately connected with other combinatorial objects, including translation planes, spreads, partial geometries, codes, generalized hexagons, and Kerdock sets (see [8, 9, 16] for example). Interestingly enough, many of the known ovoids are in fact 2-*transitive*—that is, they admit a 2-transitive automorphism group (the notion of the automorphism group is made precise in (2.3)). For example, the Suzuki groups $Sz(q)$ and the Ree groups ${}^2G_2(q)$ act 2-transitively on ovoids in 4-dimensional symplectic geometry and 7-dimensional orthogonal geometry, respectively. Furthermore, the unitary groups $PSU_3(q)$ (for suitable prime powers q) and the linear groups $PSL_2(q^3)$ (with q even) act 2-transitively on ovoids in 7- or 8-dimensional orthogonal geometry. The occurrence of such a large number of 2-transitive ovoids suggests that a classification of them is worthwhile, much in the same spirit as Kantor's classification of the finite linear spaces whose automorphism group acts 2-transitively on points [10]. The classification of the 2-transitive ovoids appears as our Main Theorem in Section 2. We discover no new ovoids, however we obtain some new results concerning the number of *isomorphism classes* of 2-transitive ovoids (see Section 2). Our proof relies on the classification of the finite 2-transitive permutation groups, which in turn relies on the recent classification of finite simple groups. We also draw upon several facts from the modular representation theory of finite groups.

2. NOTATION AND THE STATEMENT OF RESULTS

Throughout this paper, V denotes a finite classical polar space of dimension n over the finite field $\mathbb{F}$. We write

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$$\mathbb{F} = \begin{cases} GF(q) & \text{if } V \text{ is a symplectic or othogonal space} \\ GF(q^2) & \text{if } V \text{ is a unitary space,} \end{cases} \tag{2.1}$$

where $q = p^e$ and p is prime. The vector space V supports a symplectic, symmetric, or unitary form $f(\ ,\ )$: $V \times V \to \mathbb{F}$, and when V is an orthogonal space, V also supports a quadratic form Q: $V \to \mathbb{F}$. When V is symplectic or unitary we put

$$\begin{aligned} G &= \{ g \in GL(V) \mid f(g(v), g(w)) = f(v, w)\ \forall\, v, w \in V \}, \\ \Gamma &= \{ g \in \Gamma L(V) \mid f(g(v), g(w)) = \tau(g)\, f(v, w)^{\sigma(g)}\ \forall\, v, w \in V, \\ &\qquad \text{where } \tau(g) \in \mathbb{F}^* \text{ and } \sigma(g) \in \mathrm{Aut}(\mathbb{F}) \text{ depend only on } g \}. \end{aligned} \tag{2.2}$$

When V is orthogonal we have similar definitions of G and Γ, with $Q(\ )$ replacing $f(\ ,\ )$. Thus G is the isometry group of f or Q, and so it is either a symplectic group $Sp_n(q)$ (n even), an orthogonal group $O_n(q)$ (n odd), $O_n^+(q)$ (n even and Q hyperbolic) or $O_n^-(q)$ (n even and Q elliptic), or a unitary group $U_n(q)$. The group Γ is the full semilinear subgroup associated with f or Q, that is, the subgroup of $\Gamma L(V)$ which preserves the symplectic or unitary polarity f or the quadric Q. Now let $\Gamma_{\{\mathcal{O}\}}$ and $\Gamma_{(\mathcal{O})}$ be the set-wise and point-wise stabilizers of $\mathcal{O}$ in Γ, respectively. Then the *automorphism group* of $\mathcal{O}$ is defined as

$$A = \mathrm{Aut}(\mathcal{O}) = \Gamma_{\{\mathcal{O}\}} / \Gamma_{(\mathcal{O})}. \tag{2.3}$$

Loosely speaking, Γ is the "largest" group associated with the polar space V and so (2.3) gives a rather wide definition of the automorphism group.

Two ovoids are said to be *isomorphic* if there is an element of Γ taking one to the other. Clearly isomorphism is an equivalence relation and the equivalence classes will be called *isomorphism classes*. When the maximal totally singular subspaces of V have dimension 1, then the set of singular points in V is automatically an ovoid, and we call it a *trivial ovoid*. They arise in the rank 1 geometries $O_2^+(q)$, $O_3(q)$, $O_4^-(q)$, $Sp_2(q)$, $U_2(q)$, and $U_3(q)$. Furthermore, note that ovoids in certain geometries automatically give rise to ovoids in larger geometries. For instance, ovoids in $O_{2m+1}(q)$ (or in $O_{2m}^-(q)$) yield ovoids in $O_{2m+2}^+(q)$, and ovoids in $U_{2m-1}(q)$ yield ovoids in $U_{2m}(q)$. When this occurs, we say that the ovoid in the larger geometry is *induced from* the ovoid in the smaller geometry.

Equipped with the preceding notation and terminology, we are now ready to state the main result of this paper.

MAIN THEOREM. *Let V be a finite classical polar space and G the corresponding group of isometries, as above. Let $\mathcal{O}$ be an ovoid in V and assume that* $\mathrm{Aut}(\mathcal{O})$ *acts 2-transitively on the points in $\mathcal{O}$. Then G and* $\mathrm{Aut}(\mathcal{O})$

TABLE I

The 2-Transitive Ovoids

G	$\mathrm{Aut}(\mathcal{O})$	Remarks	Reference
$O_2^+(q)$	$\mathbb{Z}_2$	Trivial	
$O_3(q)$, $Sp_2(q)$, $U_2(q)$	$\mathrm{Aut}(PSL_2(q))$	Trivial	
$O_4^-(q)$	$\mathrm{Aut}(PSL_2(q^2))$	Trivial	
$U_3(q)$	$\mathrm{Aut}(PSU_3(q))$	Trivial	
$O_4^+(q)$	$\mathrm{Aut}(PSL_2(q))$	Induced from $O_3(q)$	
$Sp_4(q)$, $O_5(q)$, $O_6^+(q)$	$\mathrm{Aut}(PSL_2(q^2))$	Induced from $O_4^-(q)$, if $G = Sp_4(q)$ then q even	
$U_4(q)$	$\mathrm{Aut}(PSU_3(q))$	Induced from $U_3(q)$	
$Sp_4(q)$	$\mathrm{Aut}(Sz(q))$	q an odd power of 2, $q \geqslant 8$	[3, pp. 46ff.]
$O_5(q)$, $O_6^+(q)$	$\mathrm{Aut}(Sz(q))$	q an odd power of 2, $q \geqslant 8$, induced from $Sp_4(q)$	
$* \, O_4^+(q)$	$\mathrm{Aut}(PSL_2(q))$	$\left[\frac{\log_p(q)}{2}\right]$ isomorphism classes	
$O_7(q)$	$Sp_6(2)$	$q = 3$	[7, Sect. 3]
	$\mathrm{Aut}({}^2G_2(q))$	q an odd power of 3, $q \geqslant 27$	[16, Sect. 6], [7, Sect. 6]
	$\mathrm{Aut}(PSU_3(q))$	q a power of 3, $q \geqslant 9$	[7, Sect. 4]
$O_8^+(q)$	S_9	$q = 2$	[7, Sect. 3]
	$Sp_6(2)$	$q = 3$, induced from $O_7(3)$	
	$\mathrm{Aut}(PSU_3(q))$	$5 \leqslant q \equiv 0, 2 \pmod 3$, induced from $O_7(q)$ if $3 \mid q$	[7, Sect. 4]
	$\mathrm{Aut}(PSL_2(q^3))$	q even, $q \geqslant 4$	[7, Sect. 7]
	$\mathrm{Aut}({}^2G_2(q))$	q an odd power of 3, $q \geqslant 27$, induced from $O_7(q)$	

appear in Table I. Each entry in the table corresponds to a unique isomorphism class, except in the row marked with the symbol $*$.

The rest of this paper is devoted to the proof of the Main Theorem.

3. Preliminaries and Reductions

If $\mathcal{O}$ is an ovoid in V, then the size of $\mathcal{O}$ is given by

$$|\mathcal{O}| = \frac{\text{no. of maximal singular subspaces in } V}{\text{no. of maximal singular subspaces containing a given point in } \mathcal{O}}$$

and thus $|\mathcal{O}| = q^t + 1$, where t is given in Table II below. Table II also gives the order of the isometry group G.

TABLE II

G	Condition on n	t	$\lvert G \rvert$
$Sp_n(q)$	n even	$\frac{n}{2}$	$q^{t^2} \prod_{i=1}^{t} (q^{2i}-1)$
$O_n(q)$	n odd	$\frac{n-1}{2}$	$2q^{t^2} \prod_{i=1}^{t} (q^{2i}-1)$
$O_n^+(q)$	n even	$\frac{n}{2}-1$	$2q^{t(t+1)}(q^{t+1}-1) \prod_{i=1}^{t} (q^{2i}-1)$
$O_n^-(q)$	n even	$\frac{n}{2}$	$2q^{t(t-1)}(q^t+1) \prod_{i=1}^{t-1} (q^{2i}-1)$
$U_n(q)$	n even	$n-1$	$q^{t(t+1)/2} \prod_{i=1}^{t+1} (q^i-(-1)^i)$
$U_n(q)$	n odd	n	$q^{t(t-1)/2} \prod_{i=1}^{t} (q^i-(-1)^i)$

It is by no means the case that every classical polar space admits an ovoid. Indeed, there is evidence to suggest that very few such spaces possess an ovoid. Most of this evidence is captured in the following Proposition, which is proved in [6, 16].

PROPOSITION 1. *The following geometries do not admit an ovoid.*

(i) *$Sp_n(q)$ with q odd and $n \geqslant 4$, or with $n \geqslant 6$.*

(ii) *$O_n(q)$ with n odd, q even, and $n \geqslant 7$.*

(iii) *$O_n^-(q)$ with $n \geqslant 6$.*

(iv) *$U_n(q)$ with n odd and $n \geqslant 5$.*

(v) *$O_n^+(2)$ with $n \geqslant 10$.*

PROPOSITION 2. *The following geometries admit a unique (up to isomorphism) ovoid: $O_4^+(2)$, $O_4^+(3)$, $O_4^+(4)$, $Sp_4(2)$, $O_5(2)$, $O_6^+(2)$, $O_7(3)$, $O_8^+(2)$, and $O_8^+(3)$. Moreover all of these ovoids are 2-transitive and appear in Table* I.

Proof. This may be verified with some easy calculations, apart from $O_7(3)$, $O_8^+(2)$, and $O_8^+(3)$. For these cases, see [7, Sect. 3]. ∎

In view of Propositions 1 and 2, we can assume hereafter that

G is not one of the geometries appearing in Propositions 1 or 2, nor is G a rank 1 geometry $O_2^+(q)$, $O_3(q)$, $O_4^-(q)$, $Sp_2(q)$, $U_2(q)$, $U_3(q)$. (3.1)

Suppose for the moment that $\mathcal{O}$ is induced from a subgeometry G^* (see Sect. 2) and write $\mathcal{O}^*$ for $\mathcal{O}$ regarded as an ovoid in G^*. Then it is not difficult to show that the automorphism group of $\mathcal{O}^*$ (in the geometry G^*) acts on $\mathcal{O}^*$ in the same way that the automorphism group of $\mathcal{O}$ (in the geometry G) acts on $\mathcal{O}$. Thus we may reduce to the case in which

$$\mathcal{O} \text{ is non-induced,} \tag{3.2}$$

by which we mean $\mathcal{O}$ is not induced from any subgeometry.

Consider now the group $Z(GL(V))$ of scalars in $GL(V)$. It is obvious that $Z(GL(V)) \leqslant \Gamma_{(\mathcal{O})}$, and using (3.2) it can in fact be shown that $Z(GL(V)) = \Gamma_{(\mathcal{O})}$. Thus $A \leqslant P\Gamma$, where $P\Gamma$ denotes the projective group $\Gamma/Z(GL(V))$, and hence

$$A = P\Gamma_{\{\mathcal{O}\}}. \tag{3.3}$$

Evidently the derived group $\Gamma L(V)'$ lies in $GL(V)$ and so

$$A' \leqslant P\Gamma' \leqslant PGL(V) = GL(V)/Z(GL(V)). \tag{3.4}$$

Due to the classification of finite simple groups, all finite 2-transitive permutation groups are known. Thus we can list the possibilities for A in Table III, below. Note that Table III contains only those groups whose degree can be of the form $q^t + 1$.

Remark. In view of [4, Theorem 2], we do not need to invoke the classification of finite simple groups when q is even.

The groups in the bottom part of Table III can be discarded rather easily. For example, if A is M_{11} or M_{12}, then $(q, t) = (11, 1)$, which means $G = O_4^+(11)$. But this is impossible, for M_{11} is not involved in $O_4^+(11)$. Similarly, $S \not\cong M_{24}$. Next suppose that $A \cong \mathrm{Aut}(PSL_2(8))$ and $m = 28$. In

TABLE III

A	Comment
$PSL_2(q^t) \trianglelefteq A \leqslant P\Gamma L_2(q^t)$	$q^t \geqslant 4$
$PSU_3(q^{t/3}) \trianglelefteq A \leqslant \mathrm{Aut}(PSU_3(q^{t/3}))$	$q^{t/3} \geqslant 3$
$Sz(q^{t/2}) \trianglelefteq A \leqslant \mathrm{Aut}(Sz(q^{t/2}))$	$q^{t/2}$ an odd power of 2, $q^t \geqslant 8$
${}^2G_2(q^{t/3}) \trianglelefteq A \leqslant \mathrm{Aut}({}^2G_2(q^{t/3}))$	$q^{t/3}$ an odd power of 3, $q^{t/3} \geqslant 27$
$A_m \trianglelefteq A \leqslant S_m$	$m = q^t + 1 \geqslant 5$
$A = Sp_{2d}(2)$	$d \geqslant 3$, $q^t + 1 = 2^{d-1}(2^d \pm 1)$
$A \leqslant AGL_d(r)$	$r^d = q^t + 1$, r prime
$A = {}^2G_2(3) \cong \mathrm{Aut}(PSL_2(8))$	$q^t + 1 = 28$
$A = M_{24}$	$q^t + 1 = 24$
$A = M_{12}$	$q^t + 1 = 12$
$A = M_{11}$	$q^t + 1 = 12$

view of (3.1), G is either $O_4^+(27)$ or $U_4(3)$. However $PSL_2(8)$ is involved in neither of these.

Hereafter we can assume that A is in one of the seven infinite families of 2-transitive groups given at the top of Table III. We consider these possibilities in turn.

Case $A \leqslant AGL_d(r)$. Here A contains a normal, elementary abelian subgroup E of order r^d (with r prime) and A/E acts faithfully and transitively on the $r^d - 1$ non-trivial elements of E.

In this paragraph assume that q is odd. Thus $r = 2$ and $e = t = 1$ (recall $e = \log_p(q)$), which means $G = O_4^+(p)$. Evidently A contains a subgroup P of order p which cyclically permutes the p non-trivial elements of E and so EP is a Frobenius group of order $2^d(2^d - 1) = (p+1)\,p$. Furthermore $O^2(EP) = EP$ (recall $O^2(Y)$ is the subgroup of Y generated by all its elements of odd order). Consequently $EP = O^2(EP) \leqslant O^2(P\Gamma) \cong P\Omega_4^+(p) \cong PSL_2(p) \times PSL_2(p)$. Since an elementary abelian 2-subgroup of $PSL_2(p)$ has order 4, and since $|E| = 2^d$, we have $d \leqslant 4$. Moreover $d \neq 2$ in view of (3.1) and so we are left with the case $d = 3$. But this forces the Frobenius group $2^3 : 7$ to be a subgroup of $PSL_2(7) \times PSL_2(7)$, which is not so.

Next take q even. Evidently $d \leqslant 2$, and if $d = 2$, then $(r, d, q, t) = (3, 2, 2, 3)$, and so by (3.1) we have $G = U_4(2)$. This situation leads to an ovoid induced from $U_3(2)$, which violates (3.3). This leaves the case in which $d = 1$ and $r = q^t + 1$. Let a be an element of A of order r. Since $|P\Gamma L(V) : PGL(V)| \leqslant 2e = 2\log_2(q) < r$, we have $a \in PGL(V)$. And because a is $P\Gamma$-conjugate to a^i for all $i = 1, \ldots, r-1$, a preimage of a in $GL(V)$ has at least $r - 1$ eigenvalues. But then $2t + 2 \geqslant n \geqslant r - 1 = q^t$, forcing $(q, t) \in \{(2, 1), (2, 2), (4, 1)\}$, against (3.1).

Case $A \cong Sp_{2d}(2)$ $(d \geqslant 3)$. Here $q^t + 1 = 2^{d-1}(2^d \pm 1)$, and since $Sp_6(2)$ is not involved in $O_4^+(q)$, we have $t \geqslant 2$. But now an easy argument shows that $q = t = 3$, and so as above, (3.1) forces $G = U_4(3)$. However $Sp_6(2)$ is not involved in $U_4(3)$ and so this case cannot arise.

Case $A_m \trianglelefteq A \leqslant S_m$. $(q^t + 1 = m \geqslant 5)$. Using Lagrange's Theorem, Table II, and (3.1), we find that the only case where $|A|$ divides $|G|$ is when $(q, t) = (5, 1)$. But A_6 is not involved in $O_4^+(5)$.

We have now reduced to the case in which A appears in one of the top four rows of Table III. Thus A has a non-abelian simple normal subgroup S isomorphic to $PSL_2(q^t)$, $Sz(q^{t/2})$, $PSU_3(q^{t/3})$ or ${}^2G_2(q^{t/3})$, and

$$S \trianglelefteq A \leqslant \mathrm{Aut}(S). \tag{3.5}$$

Thus $S \leqslant PGL(V)$ in view of (3.4) and we will exploit this fact in the remaining sections.

We conclude this section by making one further reduction, namely

$$G \text{ is not } O_5(q) \text{ or } O_6^+(q). \tag{3.6}$$

For assume that G is $O_5(q)$ or $O_6^+(q)$. Then $t=2$ and because neither ${}^2G_2(q^{2/3})$ nor $PSU_3(q^{2/3})$ is involved in G, we know that S is $L_2(q^2)$ or $Sz(q)$. Now ovoids in $O_5(q)$ induce ovoids in $O_6^+(q)$ and ovoids in $O_6^+(q)$ give rise to translation planes of order q^2 via the Klein correspondence. Thus S acts faithfully on a translation plane $\mathscr{T}$. If $S \cong Sz(q)$, then it follows from [14] that $\mathscr{T}$ is a Lüneburg plane, which means $\mathcal{O}$ is induced from $Sp_4(q)$. Similarly, if $S \cong PSL_2(q^2)$, then by [15] $\mathscr{T}$ is Desarguesian, which means $\mathcal{O}$ is induced from $O_4^-(q)$. In both cases we violate (3.2) and thus we have established (3.6).

4. Some Modular Representation Theory of S

Let W be a vector space of dimension m over $\mathbb{F} = GF(q)$. Write $GL_m(q)$, $GL(W)$, or $GL_m(W, q)$ for the general linear group of W over $\mathbb{F}$, and $PGL_m(q)$, $PGL(W)$, or $PGL_m(W, q)$ for the corresponding projective group. In this section we are concerned with *p-modular projective representations* of S; that is, homomorphisms ρ from S to $PGL_m(W, q)$ for some m, q, and W. (We emphasize the fact that the underlying vector space associated with $PGL_m(W, q)$ has vector space dimension m and not $m+1$.) Of course, S does not act on the vectors of W, but S does act on the subspaces of W. Thus ρ is said to be *irreducible* if there is no proper, non-zero S-invariant subspace of W. Observe that $PGL_m(W, q)$ is contained naturally in $PGL_m(W \otimes \bar{\mathbb{F}}, \bar{\mathbb{F}})$, where $\bar{\mathbb{F}}$ is an algebraic closure of $GF(q)$. Thus ρ gives rise to a projective representation $\bar{\rho}$ of S to $PGL_m(W \otimes \bar{\mathbb{F}}, \bar{\mathbb{F}})$, and ρ is said to be *absolutely irreducible* if $\bar{\rho}$ is also irreducible.

These first two results are rather easy.

LEMMA 3. *If S has an irreducible representation in $PGL_m(q^b)$, then there is a divisor d of m such that S has an absolutely irreducible representation in $PGL_{m/d}(q^{bd})$.*

Proof. See [5, Theorem 9.21]. ∎

LEMMA 4. *Suppose that $S \leqslant PGL(W)$ and that S preserves a non-degenerate bilinear or unitary form on W. Further assume that S fixes a subspace U of W and that S is irreducible on U. Then*

(i) *U is either totally singular or non-singular,*

(ii) *S also fixes a subspace of dimension* $\dim(W) - \dim(U)$.

Proof. Clear. ▮

Now define the integers k and l according to

$$(k, l) = \begin{cases} (2, 1) & \text{if } S \cong PSL_2(q^t) \\ (4, 2) & \text{if } S \cong Sz(q^{t/2}) \\ (3, 3) & \text{if } S \cong PSU_3(q^{t/3}) \\ (7, 3) & \text{if } S \cong {}^2G_2(q^{t/3}) \end{cases} \tag{4.1}$$

and set

$$t_0 = \frac{t}{(t, l)}. \tag{4.2}$$

Then we have

LEMMA 5. (i) *If S has a non-trivial representation in $PGL_m(q^b)$, then $m \geqslant k$.*

(ii) *If the representation in* (i) *is absolutely irreducible, then*

(a) *$m \geqslant k^x$, where $x = t_0/(t_0, b)$,*

(b) *m is an* xth *power.*

Proof. Assertion (i) is presented in [12, pp. 436–437] and (ii.a) is proved in [12, Theorems 2.1, 2.2.i]. Part (ii.b) follows from the proof of Theorems 2.1 and 2.2.i in [12]. ▮

Let B be a Borel subgroup of S, so that B is a Sylow p-normalizer in S and B is the stabilizer of a point $\langle v \rangle \in \mathcal{O}$. We write

$$B = S_{\langle v \rangle} \tag{4.3}$$

and

$$\mathcal{O} = \{s(\langle v \rangle) \mid s \in S\}, \tag{4.4}$$

where $s(\langle v \rangle)$ is the image of $\langle v \rangle$ under s. Note that (3.3) and (4.4) imply

$$v \text{ lies in no } S\text{-invriant proper subspace of } V. \tag{4.5}$$

LEMMA 6. *Suppose that S has an absolutely irreducible representation in $PGL_m(W, q^b)$. Then the following hold.*

(i) *B fixes at most one point in W.*

(ii) *If $b = 1$ and $S \cong PSL_2(q)$, then B does indeed fix a point.*

Proof. Assertion (i) is a consequence of [1, Theorem 4.3(c)], and (ii) may be proved along the lines of [1, Lemma 3.1]. ▮

If $S \leqslant PGL(W)$, define $\tilde{S}$ to be the last term of the derived series of the preimage of S in $GL(W)$. Thus $\tilde{S} \leqslant GL(W)$ and $\tilde{S}$ is quasisimple. If $S \cong PSL_2(q)$, then $\tilde{S}$ is isomorphic to $SL_2(q)$ or $PSL_2(q)$, and if $S \cong PSU_3(q)$, then $\tilde{S}$ is isomorphic to $SU_3(q)$ or $PSU_3(q)$. When S is isomorphic to $Sz(q)$ or ${}^2G_2(q)$, then $\tilde{S} \cong S$. For any subgroup T of S, let $\tilde{T}$ be the preimage of T is $\tilde{S}$. Evidently $\tilde{T}$ acts on the vectors in W, and we write $\tilde{T}_w$ for the subgroup of $\tilde{T}$ fixing the vector $w \in W$.

Note that if W is an m-dimensional vector space over $GF(q^b)$, then W is also a vector space over $GF(q)$ of dimension mb. Hence there is a natural inclusion $GL_m(W, q^b) \leqslant GL_{mb}(W, q)$. We will write (W, q) and (W, q^b) for W regarded as a $GF(q)$-space and $GF(q^b)$-space, respectively; and if $w \in W$, then $wGF(q)$ and $wGF(q^b)$ denote the $GF(q)$-span and $GF(q^b)$-span of w, respectively.

LEMMA 7. *Assume that $S \leqslant PGL_{mb}(W, q)$ with $b \geqslant 2$ and that $\tilde{S} \leqslant GL_m(W, q^b)$. Further suppose that $\tilde{B}$ fixes a unique $GF(q^b)$-point $wGF(q^b)$ in (W, q^b) and that $|\tilde{B} : \tilde{B}_w| = q^b - 1$. Then B does not fix a $GF(q)$-point in (W, q).*

Proof. Suppose for a contradiction that B fixes the $GF(q)$-point $uGF(q)$ for some $u \in W$. Then obviously $\tilde{B}$ fixes the $GF(q^b)$-point $uGF(q^b)$ and so $uGF(q^b) = wGF(q^b)$ by our uniqueness assumption. Now on the one hand $\tilde{B}$ acts on the $q - 1$ non-zero vectors in $uGF(q)$, yet on the other, $\tilde{B}$ is transitive on the $q^b - 1$ non-zero vectors in $uGF(q^b)$. Hence we have reached the desired contradiction. ∎

Now we collect some information about the p-modular absolutely irreducible projective representations of $PSU_3(q)$ and ${}^2G_2(q)$.

LEMMA 8. *Assume that $S \cong PSU_3(q)$.*

(i) *The group S has an absolutely irreducible representation in $O_7(q)$ (respectively, $O_8^+(q)$), if and only if $q \equiv 0 \pmod 3$ (respectively $q \equiv 2 \pmod 3$).*

(ii) *B fixes a point $\langle w \rangle$ in the corresponding module, and $|\tilde{B} : \tilde{B}_w| = q - 1$.*

Proof. Assertion (i) follows from [11, Theorems *O*.7 and *O*.8$^+$], for example. Assertion (ii) may be established by working with the representation described explicitly in [7, pp. 598–599]. ∎

LEMMA 9. *Assume that S is isomorphic to $PSU_3(q)$ or ${}^2G_2(q)$ and that S is absolutely irreducible in $O_7(q)$ (with q a power of 3). Then B fixes a unique point in the corresponding module and this point is singular.*

Proof. First of all, [11, Theorem $O.7$] implies that $O_7(q)$ has a unique conjugacy class of absolutely irreducible subgroups isomorphic to S. Second, it is well known (see [7, Sections 4, 7]) that $O_7(q)$ has an ovoid acted on 2-transitively by an absolutely irreducible copy of S. These two statements imply that every absolutely irreducible copy of S in $O_7(q)$ acts 2-transitively on an ovoid. In particular, S does so and hence B fixes a singular point in the corresponding module. Uniqueness follows from Lemma 6.i. ∎

We now come to the main result of this section.

PROPOSITION 10. *Assume that $t \geqslant 2$, that G is unitary or orthogonal, and that S is reducible on V. Then the following hold.*

(i) *S acts irreducibly on a non-degenerate $(n-1)$-space W of V.*

(ii) *If G is orthogonal, then n is odd.*

(iii) *B fixes a non-singular point in W.*

(iv) *If G is unitary, then $S \cong PSU_3(q^{t/3})$.*

Proof. First suppose that $(q, t) = (2, 3)$. Then by (3.1), $G = U_4(2)$ and $S \cong PSL_2(8)$. However $|PSL_2(8)|$ does not divide $|U_4(2)|$, a contradiction. Therefore $(q, t) \neq (2, 3)$, and so by a theorem of Zsigmondy [17], there is a prime divisor r of $q^{2t}-1$ such that r does not divide $q^m - 1$ for $1 \leqslant m < 2t$. In particular, $r \mid q^t + 1$ and so $r \mid |S|$.

In this paragraph assume that G is unitary, so that $G = U_n(q) = U_{t+1}(q)$. (Recall n is even and $n = t+1$ by Table II and Proposition 1.iv.) Since r does not divide $|GL_{t-1}(q^2)|$, it follows that S has an irreducible constituent of dimension at least t. And since S is reducible (by assumption), the irreducible constituents have dimensions 1 and t. Therefore by Lemma 4 there is an irreducible S-invariant subspace W of V of dimension t, and W is non-degenerate, proving (i). Furthermore S embeds in $PSU_t(q)$, and as t is odd, neither $|PSL_2(q^t)|$ nor $|Sz(q^{t/2})|$ divides $|PSU_t(q)|$. Assume for the moment that $S \cong {}^2G_2(q^{t/3})$. Since S is irreducible on the t-space W, Lemma 3 ensures that there is a divisor d of t such that S has an absolutely irreducible representation in $PGL_{t/d}(q^{2d})$. Therefore by Lemma 5.ii.a, coupled with the fact that $t_0 = t/(t, 3)$ is odd, we obtain $t/d \geqslant 7^{t_0/(t_0, d)} \geqslant 7^{t_0/d} \geqslant 7^{t/3d}$, which is impossible. This leaves the case $S \cong PSU_3(q^{t/3})$, proving (iv). To prove (iii), write $v = w + w'$, where v is as in (4.3) and $w \in W$ and $w' \in W^\perp$. Since B fixes both W and $W^\perp$, it is obvious that B also fixes $\langle w \rangle$ and $\langle w' \rangle$. Now $w' \neq 0$ by (4.5). Therefore as $W^\perp$ is non-singular, $(w', w') \neq 0$, and hence $0 = (v, v) = (w, w) + (w', w') \neq (w, w)$. Thus B fixes the non-singular point $\langle w \rangle$, as desired.

For the rest of this proof assume that G is orthogonal, so that G is

$O_{2t+1}(q)$ or $O^+_{2t+2}(q)$. Since r does not divide $|GL_{2t-1}(q)|$, the group S has an irreducible constituent of degree at least $2t$. We now prove

$$S \text{ has an irreducible constituent of degree } 2t. \tag{4.6}$$

Assume for a contradiction that (4.6) fails. Then $G = O^+_{2t+2}(q)$ and S has an irreducible constituent of degree $2t+1$. Thus by Lemma 4, S acts irreducibly on a non-degenerate $(2t+1)$-space U. Thus q is odd (for there are no non-degenerate odd-dimensional subspaces in even characteristic), and so S is not a Suzuki group. By Lemma 3, there is a divisor d of $2t+1$ such that S has an absolutely irreducible representation in $PGL_{2t+1/d}(q^d)$. Using the fact that $(t, d) = 1$ we deduce from Lemma 5.ii.a that

$$2t+1 \geqslant \frac{2t+1}{d} \geqslant k^{t_0}.$$

And since $t \geqslant 2$ and $2t+1/d$ is a t_0th power (Lemma 5.ii.b), we conclude $S \cong PSU_3(q)$ or ${}^2G_2(q)$. Thus $t = 3$, $G = O^+_8(q)$, and U is a non-degenerate 7-space upon which S is absolutely irreducible. Thus the proof of (iii) in the previous paragraph shows that B fixes a non-singular point in U. However this violates Lemma 9 and so the proof of (4.6) is complete. Thus

$$\text{the irreducible constituents of } S \text{ on } V \text{ have dimensions } (1, 2t), (1, 1, 2t), \text{ or } (2, 2t). \tag{4.7}$$

We now argue that

$$S \text{ fixes a } 2t\text{-space}. \tag{4.8}$$

If the dimensions are $(1, 2t)$ or $(2, 2t)$, then it is clear from Lemma 4 that (4.8) holds. Hence we may assume that $n = 2t+2$ and that S fixes a subspace U of dimension 1. Suppose for the moment that U is not contained in $U^\perp$. Then $V = U \oplus U^\perp$, which means $U^\perp$ is a non-degenerate $(2t+1)$-space. Also the irreducible constituents of S on $U^\perp$ have dimensions 1 and $2t$, and so (4.8) holds in view of Lemma 4. Thus we can assume that $U \leqslant U^\perp$. According to (4.5), $v \notin U^\perp$ and so $V = U^\perp \oplus \langle v \rangle$. Define the linear transformation θ: $V \to V$ by $\theta(u + \lambda v) = u$ for $u \in U^\perp$ and $\lambda \in \mathbb{F}$, and let T be a transversal for $\tilde{B}$ in $\tilde{S}$. Now set $\hat{\theta} = (1/q^t + 1) \sum_{t \in T} t\theta t^{-1}$. Then as in the proof of Maschke's Theorem, $\ker(\hat{\theta})$ is an S-invariant 1-space not contained in U. Thus S fixes the 2-space $U \oplus \ker(\hat{\theta})$, and hence S fixes a $2t$-space, as desired.

Now let W be an S-invariant $2t$-space provided by (4.8). Since $t \geqslant 2$, it is easy to see that S must be irreducible on W and that W is non-degenerate. As in the second paragraph of this proof, write $v = w + w'$, where $w \in W$ and $w' \in W^\perp$. Since $W^\perp$ is either a point or a non-degenerate 2-space, and since $O_2^{\pm}(q)$ is solvable, $\tilde{S}$ acts trivially on $W^{\perp}$ (recall $\tilde{S}$ is defined before Lemma 7). Therefore $\{\tilde{s}(v) \mid \tilde{s} \in \tilde{S}\}$ is contained in $W \oplus \langle w' \rangle$. But obviously $\{\langle \tilde{s}(v) \rangle \mid \tilde{s} \in \tilde{S}\} = \mathcal{O}$ (see (4.4)) and so by (4.5) we conclude $V = W \oplus \langle w' \rangle$. Therefore $G = O_n(q) = O_{2t+1}(q)$, and so (i) and (ii) have been proved. Also the same argument as before shows that $\langle w \rangle$ is a non-singular B-invariant point, and the proof is now complete. ∎

COROLLARY 11. (i) *If $G = O_n^+(q)$ (with n even), then S is irreducible on V.*

(ii) *If $G = U_n(q)$ (with n even), then $S \cong PSU_3(q^{t/3})$ and S is irreducible on a non-degenerate $(n-1)$-space in V.*

Proof. Assertion (i) is immediate from Proposition 10.ii. As for (ii), it suffices (by Proposition 10.i, iv) to show that S is reducible on V. So for a contradiction assume that S is irreducible on V. Then by Lemma 3 there is a divisor d of $n = t + 1$ such that S has an absolutely irreducible representation in $PGL_{t+1/d}(q^{2d})$. Now by Lemma 5.ii.a, along with the facts that $(t, d) = 1$ and t is odd, we obtain $t + 1 \geqslant (t+1)/d \geqslant k^{t_0}$. Also $t \geqslant 3$, and so we are left only with the case $t = 3$, $S = PSU_3(q)$, and $G = U_4(q)$. However $PSU_3(q)$ has no 4-dimensional p-modular irreducible projective representation (see [12, Theorem 1.1] for instance). ∎

5. S-INVARIANT OVOIDS WITH t SMALL

In this section we classify the S-invariant ovoids under the assumption

$$t \leqslant 3. \tag{5.1}$$

Recall $e = \log_p(q)$. Also recall the definition of *non-induced*, given after (3.2).

PROPOSITION 12. *There are just $[e/2]$ isomorphism classes of non-induced ovoids in $O_4^+(q)$ which are invariant under $PSL_2(q)$. Each such ovoid has automorphism group* $\mathrm{Aut}(PSL_2(q))$.

Proof. In this proof assume that $G = O_4^+(q)$ and note that (3.1) ensures $q \geqslant 4$. Let us begin by describing the precise structure of $P\Gamma$. First of all, we have $P\Omega_4^+(q) = L \times L$, where $L \cong PSL_2(q)$, and the group $P\Gamma$ is a certain subgroup of $\mathrm{Aut}(L \times L) \cong \mathrm{Aut}(L) \wr 2 \cong (\mathrm{Aut}(L) \times \mathrm{Aut}(L)).2$ (wreath product). Elements in the subgroup $\mathrm{Aut}(L) \times \mathrm{Aut}(L)$ will be written (a, b) with $a, b \in \mathrm{Aut}(L)$. Let $x \in \mathrm{Aut}(L)$ satisfy $\langle L, x \rangle \cong PGL_2(q)$. Note that $|PGL_2(q) : PSL_2(q)| = (2, q-1)$ and so when q is even we take $x = 1$. Next, let ϕ be a generator for a group of field automorphisms of L, so that $|\phi| = e = \log_p(q)$. Finally, let $y \in \mathrm{Aut}(L \times L)$ be an involution which interchanges the two coordinates in $\mathrm{Aut}(L)\times\mathrm{Aut}(L)$—that is, $(a,b)^y=(b,a)$ for all $(a,b)\in\mathrm{Aut}(L)\times\mathrm{Aut}(L)$. With this notation, we have

$$P\Gamma=\langle L\times L, (x,1), (\phi,\phi), y\rangle. \tag{5.2}$$

The group $L\times L$ has precisely $2+|\mathrm{Aut}(L):L|=2+(2,q-1)\,e$ conjugacy classes of subgroups isomorphic to L, with representatives

$$L\times 1, \qquad 1\times L, \qquad \text{and} \qquad L_\alpha=\{(g,\alpha(g))\mid g\in L\}, \tag{5.3}$$

where α runs through $\{\phi^i, x\phi^i \mid 0\leqslant i<e\}$, which is a set of coset representatives of L in $\mathrm{Aut}(L)$. Thus to determine the number of classes of $PSL_2(q)$ in $P\Gamma$, it suffices to determine how $P\Gamma$ acts on these $2+|\mathrm{Aut}(L):L|$ classes. It is easily seen that

$$\begin{aligned} L_\alpha^{(1,x)} &\text{ is } L\times L\text{-conjugate to } L_{x\alpha}\\ L_\alpha^{y} &\text{ is } L\times L\text{-conjugate to } L_{\alpha^{-1}}\\ L_\alpha^{(\phi,\phi)} &\text{ is } L\times L\text{-conjugate to } L_\alpha \end{aligned} \tag{5.4}$$

and therefore $L\times 1$ and L_{ϕ^i}, $0\leqslant i\leqslant[e/2]$, are representatives of the $P\Gamma$-conjugacy classes of subgroups $PSL_2(q)$. Now $L\times 1$ acts on the $(q+1)^2$ singular points in V with $q+1$ orbits, each of size $q+1$, and the points in each orbit span a totally singular 2-space. Thus the $P\Gamma$-conjugates of $L\times 1$ do not fix an ovoid. Putting $\alpha=1$ in (5.3), we find that L_1 is actually the stabilizer of a non-singular point $\langle w\rangle$, and so any L_1-invariant ovoid must lie in $w^\perp$, and so must be induced from $O_3(q)$. Finally, assume that $e\geqslant 2$ and consider $L_i=L_{\phi^i}$, with $1\leqslant i\leqslant[e/2]$. Then L_i is absolutely irreducible and it follows from Lemma 6.ii that its Borel subgroup fixes a point p_i in V. Thus L_i acts 2-transitively on $\mathcal{O}_i=\{g(p_i))\mid g\in L_i\}$. The point p_i is in fact singular in V and so 2-transitivity and irreducibility implies that the points $g(p_i)$ are mutually non-orthogonal. Hence $\mathcal{O}_i$ is an ovoid. Further, Lemma 6.i implies that $\mathcal{O}_i$ is the unique L_i-invariant ovoid and so $N_{P\Gamma}(L_i)\leqslant P\Gamma_{\{\mathcal{O}_i\}}$. It can be shown using (5.4) that $N_{P\Gamma}(L_i)\cong\mathrm{Aut}(PSL_2(q))$, and so by (3.3) and (3.5) we conclude that $\mathrm{Aut}(\mathcal{O}_i)\cong\mathrm{Aut}(PSL_2(q))$, as desired. All that remains is to prove that the ovoids $\mathcal{O}_i$ are mutually non-isomorphic. Now if $g\in P\Gamma$ takes $\mathcal{O}_i$ to $\mathcal{O}_j$, then $\langle L_i^g, L_j\rangle\leqslant P\Gamma_{\{\mathcal{O}_j\}}$. However L_j is maximal in $L\times L$, and so $L_i^g=L_j$, which means $i=j$, as required. ∎

Next we handle the case $S\cong PSL_2(q^2)$, and for this it is convenient to have the following Lemma at hand. The information given in the Lemma is essentially well known and so we offer no proof.

LEMMA 13. *The group $PGL_4(W,q)$ has just two conjugacy classes of subgroups $PSL_2(q^2)$, which we call type* 1 *and type* 2. *Let S_i be a group of* *type i for* $i = 1, 2$, *and let* B_i *be a Borel subgroup of* S_i. *Then the following hold.*

(i) S_1 *is absolutely irreducible in* $PGL_4(W, q)$.

(ii) B_1 *fixes a unique point* $\langle w \rangle$ *in* W *and* $|\tilde{B}_1 : (\tilde{B}_1)_w| = q - 1$.

(iii) S_2 *is irreducible but not absolutely irreducible in* $PGL_4(W, q)$.

(iv) B_2 *does not fix a point in* W.

PROPOSITION 14. *The group* S *cannot be isomorphic to* $PSL_2(q^2)$.

Proof. Assume for a contradiction that $S \cong PSL_2(q^2)$. Since $t = 2$, it follows from Table II, (3.1), and (3.6) that $G = Sp_4(q)$, with q even. Since B fixes a point in V, it is clear that S must be of type 1 by Lemma 13.iv. Now it is well known (see [2] or [11, Theorem S.4]) that G has a unique conjugacy class of subgroups $PSL_2(q^2)$ of type 1. However, these subgroups are the stabilizers of elliptic quadrics in V, and so the S-invariant ovoid must be induced from $O_4^-(q)$, violating (3.2). ∎

PROPOSITION 15. *If* $S \cong PSL_2(q^3)$, *then the following hold.*

(i) $G = O_8^+(q)$ *with* q *even and* $q \geqslant 4$.

(ii) $\mathrm{Aut}(\mathcal{O}) \cong \mathrm{Aut}(PSL_2(q^3))$.

(iii) *There is just one isomorphism class of non-induced ovoids in* V *invariant under* $PSL_2(q^3)$.

Proof. (i) Here $t = 3$ and so by Table II and Proposition 1, G is $U_4(q)$, $O_7(q)$, or $O_8^+(q)$. However $|PSL_2(q^3)|$ does not divide $|U_4(q)|$ and hence G is orthogonal. Suppose for the moment that $G = O_7(q)$. Then by Lemma 3, S is absolutely irreducible, against Lemma 5.ii.a. Therefore $G = O_8^+(q)$ and so Corollary 11.i ensures that S is irreducible on V. Since S has no absolutely irreducible representation in $PGL_2(q^4)$ or $PGL_4(q^2)$, Lemma 3 shows that S is absolutely irreducible on V. Thus according to [11, Theorem $O.8^+$], q is even, and so $q \geqslant 4$ by (3.1). Thus (i) holds.

(ii) Since B fixes a unique point in (Lemma 6.i), it follows that S fixes a unique ovoid, and so $N_{P\Gamma}(S) \leqslant P\Gamma_{\{\mathcal{O}\}}$. However [11, Theorem $O.8^+$] shows that $N_{P\Gamma}(S) \cong \mathrm{Aut}(S)$, and so (ii) follows from (3.3) and (3.5).

(iii) According to [11, Theorem $O.8^+$], $P\Gamma$ has just one class of absolutely irreducible subgroups $PSL_2(q^3)$. And since S fixes a unique ovoid, there is at most one isomorphism class of non-induced ovoids in V invariant under $PSL_2(q^3)$. The existence of such an ovoid is exhibited in [7, Sect. 7]. ∎

LEMMA 16. *Assume that* $S \cong PSU_3(q)$.

(i) *If* $S \leqslant PGL_3(W, q^2)$ *for some 3-dimensional vector space* W *over* $GF(q^2)$, *then* B *fixes a unique* $GF(q^2)$-*point in* W.

(ii) *If* w *spans the point provided by* (i), *then* $|\tilde{B} : \tilde{B}_w| = q^2 - 1$.

(iii) S *has no absolutely irreducible representation in* $PGL_6(q)$.

Proof. Assertion (iii) follows from [11, Theorems $L.6$, $S.6$, $O.6^{\pm}$]. As for (i) and (ii), it is clear that W is the natural 3-dimensional module for $SU_3(q)$, and so (i) and (ii) can be checked with some easy calculations using 3×3 unitary matrices. ▮

PROPOSITION 17. *If* $S \cong PSU_3(q^{t/3})$ ($q^{t/3} \geqslant 3$ *and* $t \leqslant 3$), *then the following hold.*

(i) *Either* $p = 3$ *and* $G = O_7(q)$, *or* $q \equiv 2 \pmod 3$ *and* $G = O_8^+(q)$.

(ii) $\mathrm{Aut}(\mathcal{O}) \cong \mathrm{Aut}(PSU_3(q))$.

(iii) *There is a unique isomorphism class of non-induced ovoids in* V *invariant under* $PSU_3(q)$.

Proof. Since $PSU_3(q^{1/3})$ is not involved in $O_4^+(q)$ and $PSU_3(q^{2/3})$ is not involved in $Sp_4(q)$, it follows that $t = 3$. Consequently G is $U_4(q)$ $O_7(q)$, or $O_8^+(q)$, and we consider these in turn.

Case $G = U_4(q)$. By Corollary 11.ii, S fixes a non-degenerate 3-space W in V. Obviously W is the natural 3-dimensional projective module for S and so B does not fix a non-singular point in W. But this contradicts Proposition 10.iii.

Case $G = O_7(q)$. First suppose that S is reducible on V. Then by Proposition 10.i, S fixes a 6-space W. By Lemma 16.iii, S is irreducible but not absolutely irreducible on W, and regarding S as a subgroup of $PGL_6(W, q)$, we may write $S \leqslant PGL_3(W, q^2)$ (see the discussion before Lemma 7). But now Lemma 16.ii and Lemma 7 imply that B does not fix a point in W, against Proposition 10.iii. Thus S is irreducible on V. By Lemma 3, S is absolutely irreducible, and so the first part of (i) now is immediate from Lemma 8.i. The existence of a $PSU_3(q)$-invariant ovoid in $O_7(q)$ is well known (see [7, Sect. 4]). The uniqueness follows from the fact that $P\Gamma$ has a unique conjugacy class of absolutely irreducible subgroups $PSU_3(q)$ (see [11, Theorem $O.7$]) and that B fixes at most one point in W (Lemma 6.ii). Thus (iii) holds and it remains to prove (ii). As in Proposition 15, we have $N_{P\Gamma}(S) \leqslant P\Gamma_{\{\mathcal{O}\}}$, and by [11, Theorem $O.7$], we see that $N_{P\Gamma}(S) \cong \mathrm{Aut}(S)$. Thus (ii) holds in view of (3.3) and (3.5).

Case $G = O_8^+(q)$. Arguing as in the proof of Proposition 15.i, we find that S must be absolutely irreducible on V. The rest now follows as in the previous case. ▮

PROPOSITION 18. *If* $S \cong Sz(q^{t/2})$ *(q even,* $q^{t/2} \geqslant 8$, *and* $t \leqslant 3$*), then*

(i) $G = Sp_4(q)$,

(ii) $\mathrm{Aut}(\mathcal{O}) \cong \mathrm{Aut}(Sz(q))$,

(iii) *V has a unique isomorphism class of ovoids invariant under* $Sz(q)$.

Proof. Since $Sz(q^{1/2})$ is not involved in $O_4^+(q)$, it follows that $t \neq 1$. And we can eliminate the case $t = 3$ with the usual arguments using Corollary 11 and Lemmas 3 and 4. Therefore $t = 2$, and hence according to (3.6), $G = Sp_4(q)$, proving (i). It is well known (see [2] or [11, Theorem *S*.4]) that $Sp_4(q)$ contains a unique conjugacy class of subgroups $Sz(q)$, and so as before, the uniqueness of a $Sz(q)$-invariant ovoid in G follows from Lemma 6.i. The existence is well known, and therefore (iii) holds. The proof of (ii) is analogous to that of Propositions 15.ii and 18.ii. ∎

PROPOSITION 19. *If* $S \cong {}^2G_2(q^{t/3})$ $(p = 3,\ q^{t/3} \geqslant 27$, *and* $t \leqslant 3)$, *then*

(i) $G = O_7(q)$,

(ii) $\mathrm{Aut}(\mathcal{O}) \cong \mathrm{Aut}(S)$,

(iii) *V has a unique isomorphism class of ovoids invariant under* ${}^2G_2(q)$.

Proof. Since ${}^2G_2(q^{1/3})$ is not involved in $O_4^+(q)$, since $t \neq 2$ in view of (3.1) and (3.6), and since ${}^2G_2(q)$ is not involved in $U_4(q)$, we know that $t = 3$ and G is $O_8^+(q)$ or $O_7(q)$. However according to [13, Theorem 2.10], $G_2(q)$ has no irreducible representation of degree 8, and hence the same holds of ${}^2G_2(q)$. Therefore by Corollary 11.i we deduce $G = O_7(q)$, proving (i). Assertion (ii) and the uniqueness in (iii) follow from the usual arguments, invoking [11, Theorem *O*.7]. The existence in (iii) is well known. ∎

6. S-INVARIANT OVOIDS WITH t LARGE

We complete the proof of the Main Theorem by treating the case

$$t \geqslant 4. \tag{6.1}$$

Clearly (6.1), Proposition 1.i, and Table II ensure that G is not symplectic, and so

$$G \text{ is orthogonal or unitary.} \tag{6.2}$$

The goal now is to show that no ovoids arise.

Case $S \cong PSL_2(q^t)$. It follows from Corollary 11.ii that G is not unitary and hence G is $O_{2t+1}(q)$ or $O^+_{2t+2}(q)$ by (6.2).

First consider the case in which S is irreducible on V. Then by Lemma 3 there is a divisor d of n such that S has an absolutely irreducible representation in $PGL_{n/d}(q^d)$. Thus Lemma 5.ii.a yields

$$\frac{2t}{d} \geqslant \frac{n}{d} \geqslant 2^{t/(t,d)}. \tag{6.3}$$

However $(d, t) \leqslant 2$ and n/d is a $t/(t, d)$th power by Lemma 5.ii.b, and therefore (6.3) implies that $t \leqslant 3$, violating (6.1).

We are left with the case in which S is reducible on V. Thus by Proposition 10.i, S acts irreducibly on a $2t$-space W. As before, S has an absolutely irreducible representation in $PGL_{2t/d}(q^d)$ for some divisor d of $2t$, and hence $2t/d \geqslant 2^{t/(t,d)} \geqslant 2^{t/d}$. Consequently $d = t$ or $d = t/2$. First consider the case $d = t/2$. Thus we may write $S \leqslant PGL_4(W, q^{t/2})$, and so we are in the situation of Lemma 13, with $q^{t/2}$ replacing q. Since S is absolutely irreducible in $PGL_4(W, q^{t/2})$, Lemma 13.iii ensures that S is of type 1. But then Lemma 13.ii and Lemma 7 show that B does not fix a $GF(q)$-point in W, contrary to Proposition 10.iii. If $d = t$, then we may write $S \leqslant PGL_2(W, q^t)$ and a similar argument applies.

Case $S \cong Sz(q^{t/2})$. As in the case before, G is orthogonal. And since q is even, Proposition 1.ii implies that $G = O^+_n(q) = O^+_{2t+2}(q)$. Therefore Corollary 11.i ensures that S is irreducible on V and so for some divisor d of $2t+2$ there is an absolutely irreducible representation of S in $PGL_{n/d}(q^d)$. Since t_0 is odd and $d \mid 2t+2$, we have $(t_0, d) = 1$ and so we deduce from Lemma 5.ii.a that $2t+2 = n \geqslant n/d \geqslant 4^{t_0} \geqslant 4^{t/2}$, which contradicts (6.1).

Case $S \cong PSU_3(q^{t/3})$. First take S irreducible on V. Then by Corollary 11.ii and (6.2), G is orthogonal, and as usual, there is a divisor d of n such that S has an absolutely irreducible representation in $PGL_{n/d}(q^d)$. Consequently

$$\frac{2t+2}{d} \geqslant \frac{n}{d} \geqslant 3^{t_0/(t_0,d)}. \tag{6.4}$$

Now $2t+1 \leqslant n \leqslant 2t+2$, and so $(t_0, d) \leqslant 2$. So using Table II and the fact that $|S|$ divides $|G|$, it is not hard to show that (6.1) and (6.4) imply $(t, d) = (6, 2)$. Therefore $G = O^+_{14}(q)$ and S has an absolutely irreducible representation in $PGL_7(W, q^2)$. However, according to Lemma 8.ii (with q^2 replacing q) and Lemma 7, the group B does not fix a point in V, a contradiction.

Now take S reducible on V, so that Proposition 10 applies. In particular, S has an irreducible representation in $PGL_{2t/f}(q^f)$, where

$$f = \begin{cases} 1 & \text{if } G \text{ is orthogonal} \\ 2 & \text{if } G \text{ is unitary.} \end{cases}$$

As usual, there is a divisor d of $2t/f$ such that S has an absolutely irreducible representation in $PGL_{2t/df}(q^{df})$, and thus $2t/df \geqslant 3^x$, where $x = t_0/(t_0, df)$, and $2t/df$ is an xth power. Consequently $2t/df \in \{3, 6\}$. If $2t/df = 6$, then S is absolutely irreducible in $PGL_6(q^{t/3})$, violating Lemma 16.iii (with $q^{t/3}$ replacing q). Therefore $2t/df = 3$. But now Lemma 16.ii together with Lemma 7 imply that B does not fix a point in W, against Proposition 10.iii.

Case $S \cong {}^2G_2(q^{t/3})$. The argument here is similar yet easier than those in the previous three cases, so we leave it to the reader.

The proof of the Main Theorem is now complete.

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References

1. C. W. Curtis, Modular representations of finite groups with split (B, N)-pairs, *in* "Seminar on Algebraic Groups and Related Finite Groups" Lecture Notes in Mathematics, Vol. 131, Springer-Verlag, Berlin/Heidelberg/New York, 1970.
2. D. E. Flesner, Maximal subgroups of $PSp_4(2^n)$ containing central elations or noncentered skew elations, *Illinois J. Math.* **19** (1975), 247–268.
3. J. W. P. Hirschfeld, "Finite Projective Spaces of Three Dimensions," Oxford Univ. Press (Clarendon), London/New York, 1985.
4. D. F. Holt, Transitive permutation groups in which an involution central in a Sylow 2-subgroup fixes a unique point, *Proc. London Math. Soc.* (3) **37** (1978), 165–192.
5. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.
6. W. M. Kantor, Strongly regular graphs defined by spreads, *Israel J. Math.* **41** (1982), 298–312.
7. W. M. Kantor, Ovoids and translation planes, *Canad. J. Math.* **34** (1982), 1195–1207.
8. W. M. Kantor, Spreads, translation planes and Kerdock sets I, *SIAM J. Algebraic Discrete Methods* **3** (1982), 151–165.
9. W. M. Kantor, Spreads, translation planes and Kerdock sets II, *SIAM J. Algebraic Discrete Methods* **3** (1982), 308–316.
10. W. M. Kantor, Homogeneous designs and geometric lattices, *J. Combin. Theory Ser. A* **38** (1985), 66–74.
11. P. B. Kleidman, The low-dimensional finite classical groups and their subgroups, in preparation.

12. M. W. LIEBECK, On the orders of maximal subgroups of the finite classical groups, *Proc. London Math. Soc.* (3) **50** (1985), 426–446.
13. M. W. LIEBECK, The affine permutation groups of rank three, *Proc. London Math. Soc.* (3) **54** (1987), 477–516.
14. R. A. LIEBLER, A characterization of the Lüneburg planes, *Math. Z.* **126** (1972), 82–90.
15. H. LÜNEBURG, Charakterisierungen der endlichen Desarguesschen projektiven Ebenen, *Math. Z.* **85** (1964), 419–450.
16. J. A. THAS, Ovoids and spreads of finite classical polar spaces, *Geom. Dedicata* **10** (1981), 135–144.
17. K. ZSIGMONDY, Zur Theorie der Potenzreste, *Monatsh. für Math. u. Phys.* **3** (1982), 265–284.